Propoties of Cross Product (geometric)

let $\vec{u}, \vec{v} \in \mathbb{R}^3$

- $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ & $\vec{v}$
- $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$, θ is angle between $\vec{u}$ & $\vec{v}$
- $\vec{u}$ & $\vec{v}$ are parallel if & only if $\vec{u} \times \vec{v} = 0$

Example 1

$\vec{u} = \langle 5, 3, -1 \rangle \quad \vec{v} = \langle 8, 4, 2 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & -1 \\ 8 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 5 & -1 \\ 8 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 5 & 3 \\ 8 & 4 \end{vmatrix} \vec{k}$$

$$= ((3)(2) - (4)(-1))\vec{i} - ((5)(2) - (-1)(8))\vec{j} + ((5)(4) - (3)(8))\vec{k}$$

$$= 10\vec{i} - 18\vec{j} + (-4)\vec{k} = \langle 10, -18, -4 \rangle$$

Recall: Properties of Cross Product

let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3 \quad c \in \mathbb{R}$

- $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$
- $c\vec{u} \times \vec{v} = c(\vec{u} \times \vec{v}) = u \times c\vec{v}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w}$
- $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$
- $\vec{u}$ & $\vec{v}$ are both orthogonal to $\vec{u} \times \vec{v}$
- $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}|\sin\theta \quad \theta$ = angle between $\vec{u}$ & $\vec{v}$
- $\vec{u} \times \vec{v} = 0$ if & only if $\vec{u}$ & $\vec{v}$ are parallel

Example 2

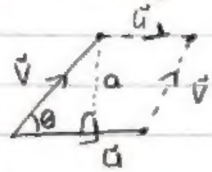
take $\vec{v} \times \vec{u}$

$\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$

$\quad = -\langle 10, -18, -4 \rangle = \langle -10, 18, 4 \rangle$

- $\vec{u} \times \vec{v}$ is computed using right hand rule

Geometry Cross Product



$\sin\theta = a/|\vec{v}|$

$a = |\vec{v}|\sin\theta$

area of parallelogram determined by $\vec{u}$ & $\vec{v}$ is $A = (\text{base})(\text{height})$

$A = |\vec{u}|a = |\vec{u}||\vec{v}|\sin\theta$

Proof of $|\vec{u}||\vec{v}|\sin\theta = \vec{u}\times\vec{v}$

- we used algebraic properties to compute

$|\vec{u}\times\vec{v}|^2 = (\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{v})$ by prop. of dot product

$= \vec{u}\cdot(\vec{v}\times(\vec{u}\times\vec{v}))$ by prop of cross product

$= \vec{u}\cdot((\vec{v}\cdot\vec{v})\vec{u} - (\vec{v}\cdot\vec{u})\vec{v})$ prop. of cross

$= (\vec{v}\cdot\vec{v})(\vec{u}\cdot\vec{u}) - (\vec{v}\cdot\vec{u})(\vec{u}\cdot\vec{v})$ prop of dot

$= |\vec{v}|^2|\vec{u}|^2 - (\vec{u}\cdot\vec{v})^2$ prop. of dot

$= (|\vec{v}||\vec{u}|)^2 - (|\vec{u}||\vec{v}|\cos\theta)^2$ geometric of dot

$= (|\vec{u}||\vec{v}|)^2 - (|\vec{u}||\vec{v}|)^2\cos^2\theta$

$= (|\vec{u}||\vec{v}|)^2(1-\cos^2\theta)$

$= (|\vec{u}||\vec{v}|)^2(\sin^2\theta)$

$= (|\vec{u}||\vec{v}|\sin\theta)^2$

$|\vec{u}\times\vec{v}|^2 = (|\vec{u}||\vec{v}|\sin\theta)^2$

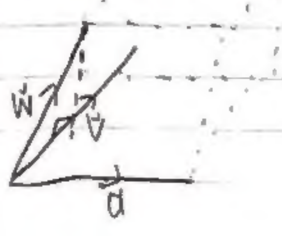
θ is angle between $\vec{u}$ & $\vec{v}$ so $\theta\in[0,\pi]$

$\sin\theta \geq 0$ on that interval

$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$

- magnitude of cross product is area of parallelogram determined by $\vec{u}$ & $\vec{v}$

- scalar triple product: $\vec{u}\cdot(\vec{v}\times\vec{w})$ is the signed volume of parallelopiped determined by $\vec{u}$, $\vec{v}$, & $\vec{w}$



← parallelopiped

12.5 Lines & Planes

- equation of line in 2-space: $ax + by - c = 0$
- in 2-space:

line is the set of points with $\vec{n} \cdot \vec{x} = c$

- generalize equation in 3-space

$\vec{n} \cdot \vec{x} = d$

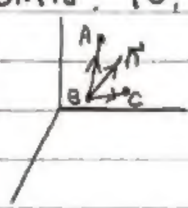
$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = d$

$ax + by + cz = d$ (plane in 3-space)

- if we knew two non-parallel vectors $\vec{u}$ & $\vec{v}$ which lie in plane (head & tail can be expressed in plane at same time) then $\vec{n} = \vec{u} \times \vec{v}$ is a normal vector to plane & it's perpendicular to every vector in plane

Ex: Find vector equation of plane

points: $(0,1,3)$, $(4,9,7)$, & $(1,2,3)$



vectors:

$\vec{u} = \langle 4-0, 9-1, 7-3 \rangle = \langle 4, 8, 4 \rangle$

$\vec{v} = \langle 1-0, 2-1, 3-3 \rangle = \langle 1, 1, 0 \rangle$

(in desired plane)

use normal vector, $\vec{n} = \vec{u} \times \vec{v}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 8 & 4 \\ 1 & 1 & 0 \end{vmatrix} = \langle -4, 4, -4 \rangle = -4\langle 1, -1, 1 \rangle$$

plane has equation

$\vec{n} \cdot \vec{x} = d$

$\langle 1, -1, 1 \rangle \cdot \langle x, y, z \rangle = d$

$x - y + z = d$

to compute d use $(0,1,3)$

$d = 0 - 1 + 3 = 2$

plane has equation:

$x - y + z = 2$